SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS (Singularity theory of smooth maps and related geometry)

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SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS

九州大学・大学院数理学研究院 佐伯 修 (OSAMU SAEKI)

Faculty of Mathematics, Kyushu University

ABSTRACT. We characterize those smooth 1-connected open 4-manifolds with certain finite type properties which admit proper special generic maps into 3-manifolds. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a proper special generic map into $\mathbf{R}^3$ if and only if it is diffeomorphic to $\mathbf{R}^4$. We also characterize those smooth 4-manifolds homeomorphic to $L\times\mathbf{R}$ for some closed orientable 3-manifold L which admit proper special generic maps into $\mathbf{R}^3$.

1. INTRODUCTION

A *special generic map* $f : M \to N$ between smooth manifolds is a smooth map with at most *definite fold singularities*, which have the normal form

$$(1.1) \qquad (x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, x_n^2 + x_{n+1}^2 + \cdots + x_m^2),$$

where $m = \dim M \geq \dim N = n$. For some typical examples of special generic maps, refer to Fig. 1. Note also that the map $\mathbf{R}^m \to \mathbf{R}^n$ defined by (1.1) is itself a proper special generic map, where a continuous map is *proper* if the inverse image of a compact set is always compact. Submersions are also considered special generic maps.

It has been known as the Reeb Theorem [19] that if a smooth connected closed m-dimensional manifold admits a special generic map into $\mathbf{R}$, then it is homeomorphic to the m-sphere S^m. In [20, 21], the author has shown that a smooth connected closed m-dimensional manifold M admits a special generic map into $\mathbf{R}^n$ for every n with $1 \leq n \leq m$ if and only if M is diffeomorphic to the standard m-sphere S^m. In [23, 24] Sakuma and the author found some pairs of homeomorphic smooth closed 4-manifolds such that one of them admits a special generic map into $\mathbf{R}^3$, while the other does not. These show that special generic maps are sensitive to detecting distinct differentiable structures on a given topological manifold.

On the other hand, it has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided $m \neq 4$ (see [15, 26]), while for $m = 4$, there exist uncountably many

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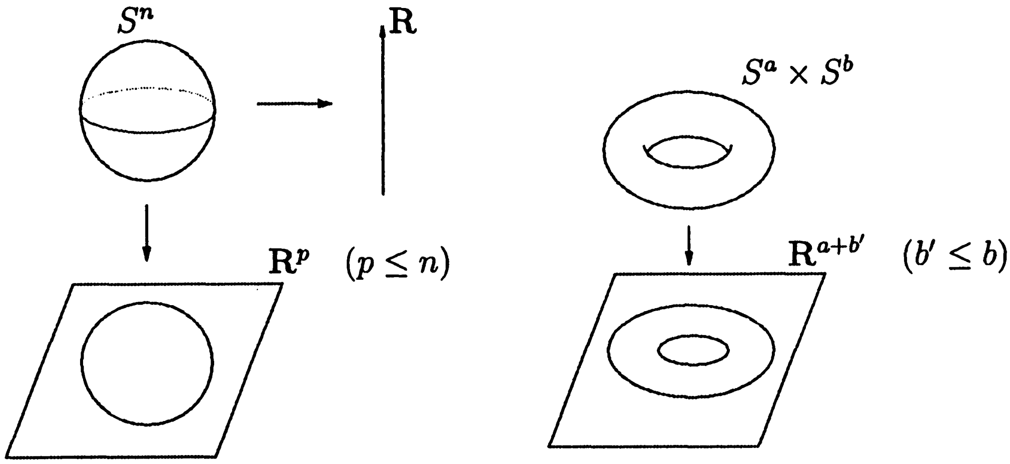

FIGURE 1. Examples of special generic maps

distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 6, 8, 27]). In fact, it is known that most open 4-manifolds admit infinitely (and very often, uncountably) many distinct differentiable structures [1, 3, 5, 7].

In this paper, we characterize those smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds, using the solution to the Poincaré Conjecture in dimension three (see [16, 17, 18] or [14], for example). Here, an open 4-manifold is of finite type if its homology is finitely generated and it has only finitely many ends, whose associated fundamental groups are stable and finitely presentable. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ is diffeomorphic to the standard $\mathbf{R}^4$ if and only if it admits a proper special generic map into $\mathbf{R}^3$.

Furthermore, we show that if a smooth 4-manifold M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and if M admits a proper special generic map into $\mathbf{R}^3$, then M is diffeomorphic to $L \times \mathbf{R}$ and the 3-manifold L admits a special generic map into $\mathbf{R}^2$.

All these results claim that among the (uncountably or infinitely) many distinct differentiable structures on a certain open topological 4-manifold, there is at most one smooth structure that allows the existence of a proper special generic map into a 3-manifold.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise indicated. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

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2. PRELIMINARIES

Let us first recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and the points x and x' belong to the same connected component of $f^{-1}(y)$. We define

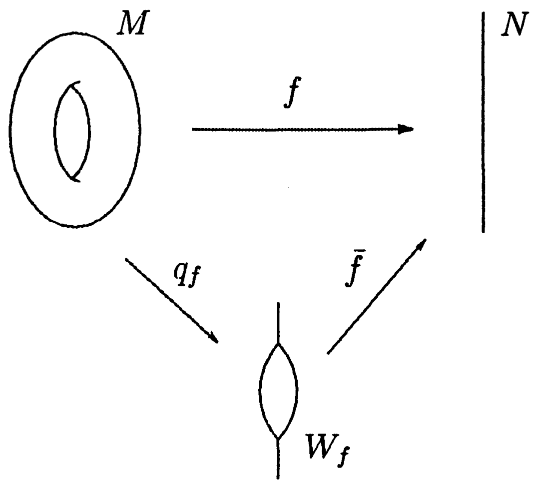

FIGURE 2. Stein factorization

$W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and denote by $q_f : M \to W_f$ the quotient map. Then we see easily that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutative. The above diagram is called the *Stein factorization* of f (see [13]). Refer to Fig. 2 for an example.

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [2, 20]).

Proposition 2.2. *Let $f : M \to N$ be a proper special generic map between smooth manifolds with $m = \dim M > \dim N = n$. Then we have the following.*

1. *The set of singular points $S(f)$ of f is a regular submanifold of M of dimension $n-1$, which is closed as a subset of M.*
2. *The quotient space W_f has the structure of a smooth n-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N$ is an immersion.*
3. *The quotient map $q_f : M \to W_f$ restricted to $S(f)$ is a diffeomorphism onto ∂W_f.*
4. *If M is connected, then the quotient map q_f restricted to $M \setminus S(f)$ is a smooth fiber bundle over $\mathrm{Int}\, W_f$. Furthermore, if $S(f) \neq \emptyset$, then the fiber is the standard $(m-n)$-sphere S^{m-n}.*

See Fig. 3 for an illustrative explanation.

Using the above proposition, the author proved the following [20].

Theorem 2.3 (Disk bundle theorem). *Let $f : M \to N$ be a proper special generic map between smooth connected manifolds with $\dim M = m$ and $\dim N = n$. If $m - n = 1, 2, 3$ and $S(f) \neq \emptyset$, then M is diffeomorphic to the boundary of a D^{m-n+1}-bundle over W_f with $O(m-n+1)$ as structure group.*

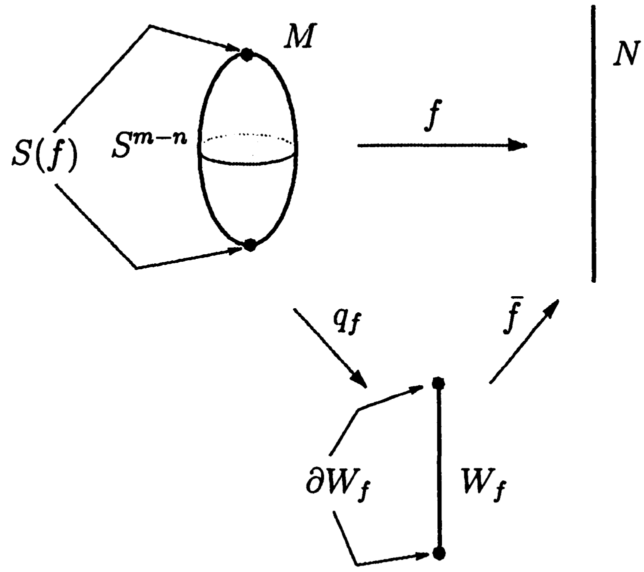

FIGURE 3. Proposition 2.2

In the following, we recall several notions concerning ends of manifolds. For details, the reader is referred to Siebenmann's thesis [25].

Definition 2.4. Let X be a Hausdorff space. Consider a collection ε of subsets of X with the following properties.

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,

(ii) If $G, G' \in \varepsilon$, then there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$,

(iii) $\bigcap\limits_{G \in \varepsilon} \overline{G} = \emptyset$.

Adding to ε every connected open non-empty set $H \subset X$ with compact frontier such that $G \subset H$ for some $G \in \varepsilon$, we produce a collection satisfying (i), (ii) and (iii), which we call the *end* of X determined by ε.

An *end* of a Hausdorff space X is a collection ε of subsets of X which is maximal with respect to the properties (i), (ii) and (iii) above.

A *neighborhood* of an end ε is any set $N \subset X$ that contains some member of ε. (See Fig. 4.)

Definition 2.5. Let ε be an end of a topological manifold X. The fundamental group π_1 is *stable* at ε if there exists a sequence of path connected neighborhoods of ε, $X_1 \supset X_2 \supset \cdots$, with $\bigcap \overline{X}_i = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \cdots .$$

The following lemma is proved in [25].

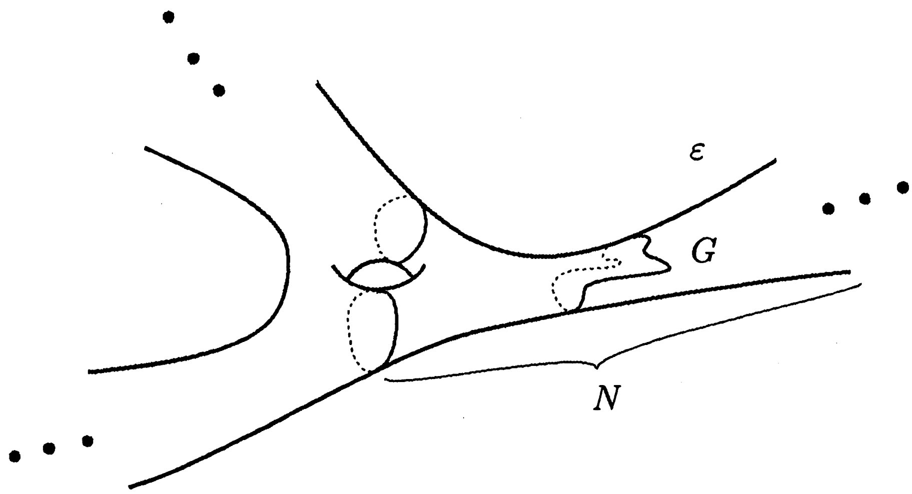

FIGURE 4. Ends of a manifold

Lemma 2.6. *If π_1 is stable at ε and $Y_1 \supset Y_2 \supset \cdots$ is any path connected sequence of neighborhoods of ε such that $\bigcap \overline{Y}_i = \emptyset$, then for any choice of base points and base paths, the inverse sequence*

$$\mathcal{G}: \qquad \pi_1(Y_1) \xleftarrow{g_1} \pi_1(Y_2) \xleftarrow{g_2} \cdots$$

induced by the inclusions is stable, i.e. there exists a subsequence

$$\pi_1(Y_{i_1}) \xleftarrow{h_1} \pi_1(Y_{i_2}) \xleftarrow{h_2} \cdots$$

inducing isomorphisms

$$\mathrm{Im}(h_1) \xleftarrow{\cong} \mathrm{Im}(h_2) \xleftarrow{\cong} \cdots,$$

where each h_j is a suitable composition of g_i's.

Definition 2.7. When π_1 is stable at an end ε, we define $\pi_1(\varepsilon)$ to be the projective limit $\varprojlim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above. According to [25], $\pi_1(\varepsilon)$ is well defined up to isomorphism.

Let us introduce the following definition.

Definition 2.8. An open manifold M is of *finite type* if

(i) M has finitely many ends,
(ii) for each end ε, π_1 is stable at ε with $\pi_1(\varepsilon)$ being finitely presentable, and
(iii) $H_*(M; \mathbf{Z}_2)$ is finitely generated.

We will need the following result due to Husch–Price [11, 12].

Lemma 2.9 (Husch–Price, 1970). *Let W be an open orientable 3-manifold of finite type. Then there exists a compact orientable 3-manifold $\widetilde{W}$ and an embedding $h : W \to \widetilde{W}$ such that $h(\mathrm{Int}\, W) = \mathrm{Int}\, \widetilde{W}$.*

3. Open 4-manifolds that admit special generic maps

In the following, a manifold is *open* if it has no boundary and each of its component is non-compact, while a manifold is *closed* if it has no boundary and is compact.

Theorem 3.1. *Let M be a smooth 1-connected open 4-manifold of finite type. Then there exists a proper special generic map $f : M \to N$ into a smooth 3-manifold N with $S(f) \neq \emptyset$ if and only if M is diffeomorphic to the connected sum of a finite number of copies of the following 4-manifolds:*

(1) $\mathbf{R}^4$,
(2) *the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$,*
(3) *the total space of a 2-plane bundle over S^2,*
(4) *the total space of an S^2-bundle over S^2,*

where at least one manifold of the form (1), (2) *or* (3) *should appear in the connected sum.*

Sketch of proof. Let $f : M \to N$ be a proper special generic map into a 3-manifold N. Then we can prove that the quotient space W_f in the Stein factorization of f is an open 3-manifold of finite type. Since M is 1-connected, so is W_f. By the solution to the Poincaré Conjecture together with the Husch–Price Lemma (Lemma 2.9), we see that $W_f \cong D^3 \setminus F$ or $\natural^k(S^2 \times [0,1]) \setminus F$, where F is a compact surface (possibly with boundary) contained in the boundary. On the other hand, M is diffeomorphic to the boundary of a D^2-bundle over W_f by the Disk bundle theorem, Theorem 2.3. Then we easily get the desired conclusion.

Conversely, it is easy to construct explicitly a proper special generic map into a 3-manifold for each 4-manifold in the list. □

Remark 3.2. Every 4-manifold as in Theorem 3.1 admits infinitely many (or uncountably many) distinct smooth structures. Theorem 3.1 implies that among them there is exactly one structure that allows the existence of a proper special generic map into a 3-manifold.

In particular, we have the following.

Corollary 3.3. *Let M be a smooth 4-manifold homeomorphic to $\mathbf{R}^4$. Then there exists a proper special generic map $f : M \to \mathbf{R}^3$ if and only if M is diffeomorphic to the standard $\mathbf{R}^4$.*

We also have the following[1].

Theorem 3.4. *Let L be a smooth connected closed orientable 3-manifold. A smooth 4-manifold M homeomorphic to $L \times \mathbf{R}$ admits a proper special generic map into $\mathbf{R}^3$ if and only if M is diffeomorphic to $L \times \mathbf{R}$ and L is a smooth closed 3-manifold that admits a special generic map into $\mathbf{R}^2$.*

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[1]Theorem 3.4 was first conjectured by Kazuhiro Sakuma to whom the author would like to express his sincere gratitude.

Sketch of proof. Suppose M is homeomorphic to $L \times \mathbf{R}$ and let $f : M \to N$ be a proper special generic map into a 3-manifold N. Then one can show that W_f is of finite type and has exactly two ends $F_i \times [0, \infty)$, $i = 1, 2$, for some surfaces F_i. Furthermore, the inclusions $F_i \times \{0\} \hookrightarrow W_f$ induce isomorphisms of fundamental groups. By the standard theory of 3-manifolds together with the solution to the Poincaré Conjecture and the Husch–Price Lemma, we see that $W_f \cong (F_1 \times \mathbf{R})\sharp(\sharp^k D^3)$ (for example, see [10]). Since M is homeomorphic to $L \times \mathbf{R}$, we see that $W_f \cong F_1 \times \mathbf{R}$. Therefore, M is diffeomorphic to $L' \times \mathbf{R}$ for some 3-manifold L'. Note that $\pi_1(L') \cong \pi_1(L)$ is free. Therefore, $L' \cong L \cong \sharp^\ell(S^1 \times S^2)$, and hence there exists a special generic map $g : L \to \mathbf{R}^2$ by a result of Burlet–de Rham [2].

Conversely, if L admits a special generic map $g : L \to \mathbf{R}^2$, then

$$g \times \mathrm{id}_{\mathbf{R}} : L \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$$

is a proper special generic map, where $\mathrm{id}_{\mathbf{R}}$ denotes the identity map of $\mathbf{R}$. □

Conjecture 3.5. Let M be a topological 4-manifold. Then there exists at most one smooth structure on M that allows the existence of a proper special generic map into $\mathbf{R}^3$.

Remark 3.6. In the above conjecture, the *properness* of the special generic map is essential. Let $f : M \to N$ be a special generic map of an open 4-manifold and assume that M' is homeomorphic to M. Then there exists a "formal solution" over M' on the jet level for the open differential relation corresponding to special generic maps. Therefore, M' admits a special generic map by the Gromov h-principle for open manifolds [9]. Note that even if f is proper, the resulting special generic map on M' may not be proper.

Compare this with the following: if a smooth 4-manifold M is homeomorphic to $\mathbf{R}^4$, then there exists a proper special generic map $g : M \to \mathbf{R}^4$. In the equidimensional case, the C^0 dense h-principle holds and the properness can be preserved (see [9]).

References

[1] Ž. Bižaca and J. Etnyre, *Smooth structures on collarable ends of 4-manifolds*, Topology **37** (1998), 461–467.

[2] O. Burlet and G. de Rham, *Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan*, Enseignement Math. (2) **20** (1974), 275–292.

[3] F. Ding, *Smooth structures on some open 4-manifolds*, Topology **36** (1997), 203–207.

[4] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[5] F. Fang, *Embedding 3-manifolds and smooth structures of 4-manifolds*, Topology Appl. **76** (1997), 249–259.

[6] M.H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[7] R.E. Gompf, *An exotic menagerie*, J. Differential Geom. **37** (1993), 199–223.

[8] R.E. Gompf and A.I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, 1999.

[9] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 9, Springer-Verlag, Berlin, 1986.

[10] J. Hempel, *3-Manifolds*, Ann. of Math. Studies, No. 86, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1976.

[11] L.S. Husch and T.M. Price, *Finding a boundary for a* 3*-manifold*, Ann. of Math. (2) **91** (1970), 223–235.

[12] L.S. Husch and T.M. Price, *Addendum to: "Finding a boundary for a* 3*-manifold"*, Ann. of Math. (2) **93** (1971), 486–488.

[13] H. Levine, *Classifying immersions into* $\mathbf{R}^4$ *over stable maps of* 3*-manifolds into* $\mathbf{R}^2$, Lecture Notes in Math., Vol. 1157, Springer-Verlag, Berlin, 1985.

[14] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Math. Monographs, Vol. 3, Amer. Math. Soc., Providence, RI; Clay Math. Institute, Cambridge, MA, 2007.

[15] J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

[16] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint, arXiv:math/0211159v1 [math.DG].

[17] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math/0303109v1 [math.DG].

[18] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint, arXiv:math/0307245v1 [math.DG].

[19] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles 1183, Hermann, Paris, 1952, pp. 91–154.

[20] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293.

[21] O. Saeki, *Topology of special generic maps into* $\mathbf{R}^3$, Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. **5** (1993), 161–186.

[22] O. Saeki, *Special generic maps on open* 4*-manifolds*, J. of Singularities 1 (2010), 1–12.

[23] O. Saeki and K. Sakuma, *On special generic maps into* $\mathbf{R}^3$, Pacific J. Math. **184** (1998), 175–193.

[24] O. Saeki and K. Sakuma, *Special generic maps of* 4*-manifolds and compact complex analytic surfaces*, Math. Ann. **313** (1999), 617–633.

[25] L. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Doctoral dissertation, Princeton University, 1965.

[26] J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

[27] C.H. Taubes, *Gauge theory on asymptotically periodic* 4*-manifolds*, J. Differential Geom. **25** (1987), 363–430.

Faculty of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: saeki@math.kyushu-u.ac.jp